CHAPTER 2 LOCATING ROOTS OF EQUATIONS

In this Chapter, you will learn:

- three methods to find the roots of equations,
- the difference among the three methods.

1. BISECTION METHOD

If a function $f(x)$ is *continuous* between *a* and *b*, and $f(a)$ and $f(b)$ are of *opposite signs*, then there exists at least one root between *a* and *b*.

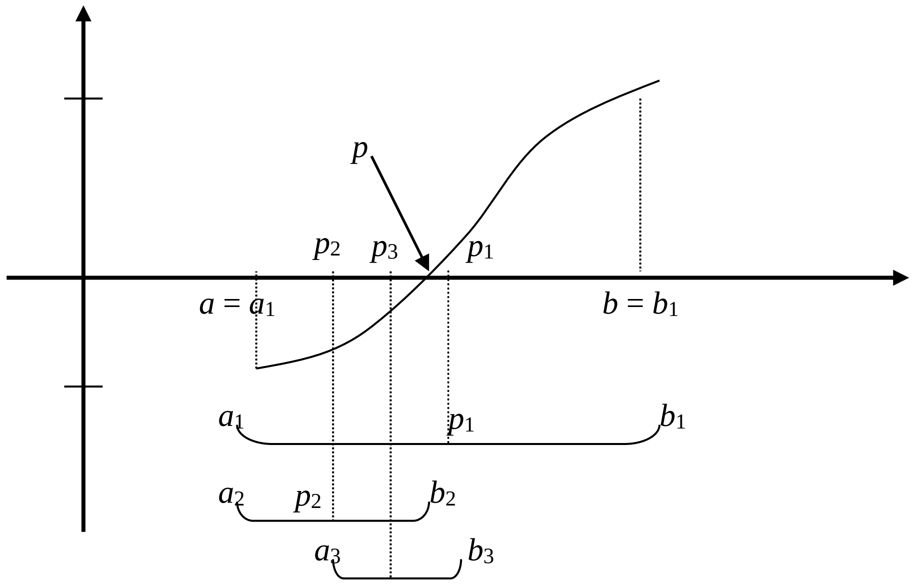

Figure 1: The graphical representation of the Bisection method

TO START the Bisection method, an interval $[a,b]$ must be found with $f(a)f(b) < 0$. Set $a_1 = a$, $b_1 = b$ and let p_1 be the midpoint of the interval $[a_1, b_1]$.

$$p_1 = \frac{a_1 + b_1}{2}$$

IN GENERAL

An interval $\left[a_{n+1}, b_{n+1}\right]$ containing an approximation to a root of $f(x)=0$ is constructed from an interval $\left[a_n, b_n\right]$ containing the root by the first letting

$$\boxed{p_n = \frac{b_n + a_n}{2}}$$

Then set $a_{n+1} = a_n$ and $b_{n+1} = p_n$, if $f(a_n)f(p_n) < 0$, and otherwise $a_{n+1} = p_n$ and $b_{n+1} = b_n$.

There are three iteration criteria:

1. If $f(a_n)$ and $f(p_n)$ have OPPOSITE signs, then p is in the interval $\left[a_n, p_n\right]$.
2. If $f(a_n)$ and $f(p_n)$ have SAME signs, then p is in the interval $\left[p_n, b_n\right]$.
3. If $f(p_n)=0$, then p_n is a ROOT of the equation $f(x)=0$.

Convergence Analysis

If the Bisection method is applied to a continuous function f on an interval $[a,b]$,where $f(a)\,f(b)<0$, then after n steps an approximated root will have been computed with error at most $\frac{b-a}{2^n}$, where $n \geq 1$.

In total, there are three stopping criteria:

1. If one of the midpoints happens to COINCIDE with the root.
2. The length of search interval is LESS THAN OR EQUAL tolerance. No of iteration, n can be computed by $\frac{b-a}{2^n} \leq tolerance$
3. If the number of iterations EXCEEDS a preset bound N_0.

Example 1: Given $f(x) = x^3 + 4x^2 - 10 = 0$ starting with $a=1$, $b=2$ and tolerance $2*10^{-3}$ by Bisection method. Compute the number of required iterations to fulfill the tolerance and approximate the root.

$$\frac{b-a}{2^n} \leq 2*10^{-3}$$
$$8.96 \leq n$$
$$n = 9$$

Iteration, n	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	$f(a_n)f(p_n)$
1	1.000000	2.000000	1.500000	-5	2.375000	-
2	1.000000	1.500000	1.250000	-5	-1.796875	+
3	1.250000	1.500000	1.375000	-1.796875	0.162109	-
4	1.250000	1.375000	1.312500	-1.796875	- 0.848389	+
5	1.312500	1.375000	1.343750	- 0.848389	- 0.350983	+
6						
7						
8						
9						

2. SECANT METHOD

Although the Bisection Method always converges, the speed of convergence is usually too slow for general use. The Secant method does not have the root bracketing property of the Bisection Method. But it does converge. It generally does so much faster than the Bisection Method.

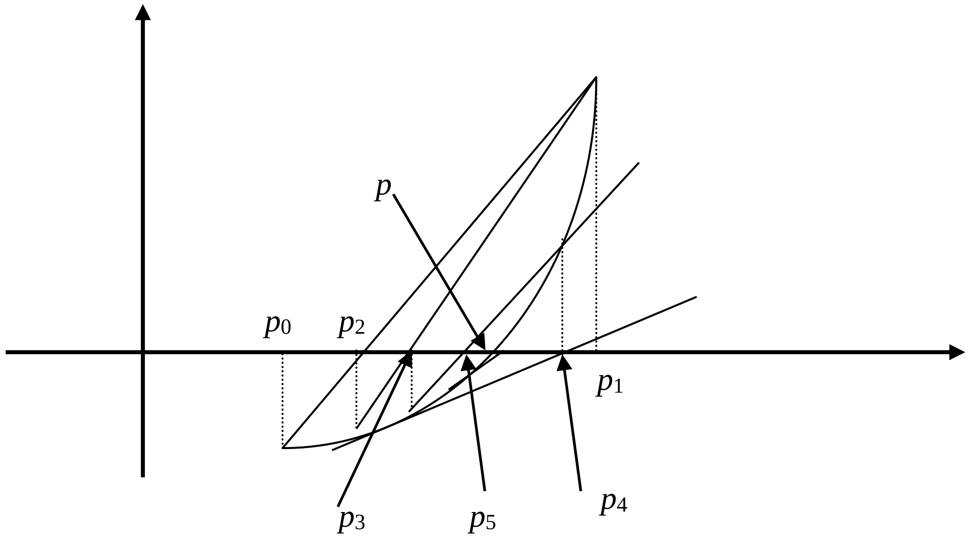

Figure 2: The graphical representation of the Secant method

IN GENERAL

The approximation p_{n+1} for $n > 1$ to a root of $f(x) = 0$ is computed from the approximations p_n and p_{n+1} using the equation.

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}}$$

There are two stopping criteria:

1. We assume that p_n is sufficiently accurate when $|p_n - p_{n-1}|$ is within a given tolerance.
2. The maximum number of iterations is given.

Example 2: Find a root of $f(x) = x^3 + 4x^2 - 10 = 0$ starting with $p_0 = 1$, $p_1 = 2$ and tolerance $1*10^{-3}$ by Secant method.

Iteration, n	p_{n+1}	$f(p_{n+1})$
1	1.263157	- 1.602288
2	1.338827	-0.430378
3		
4		
5		

3. NEWTON'S METHOD

The *Bisection* and *Secant* methods both have geometric representations that use the zero of an approximating line to the graph of a function *f* to approximate the solution to $f(x) = 0$.

The accuracy of the *Secant* method increases than the *Bisection* method because the secant line to the curve is better approximation to the graph of *f*.

The line that best approximates the graph of the function at a point is the *tangent line*.

Using this line instead of the secant line produces *Newton's* method (also called *Newton-Raphson* method).

Suppose that p_0 is an initial approximation to the root *p* of the equation $f(x) = 0$ and that *f'* exists in an interval containing all the approximations to *p*. The slope of the tangent line to the graph of *f* at the point $(p_0, f(p_0))$ is $f'(x)$, so the equation of this tangent line is

$$y - f(p_0) = f'(p_0)(x - p_0)$$

which implies that

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \text{ where } f'(p_0) \neq 0$$

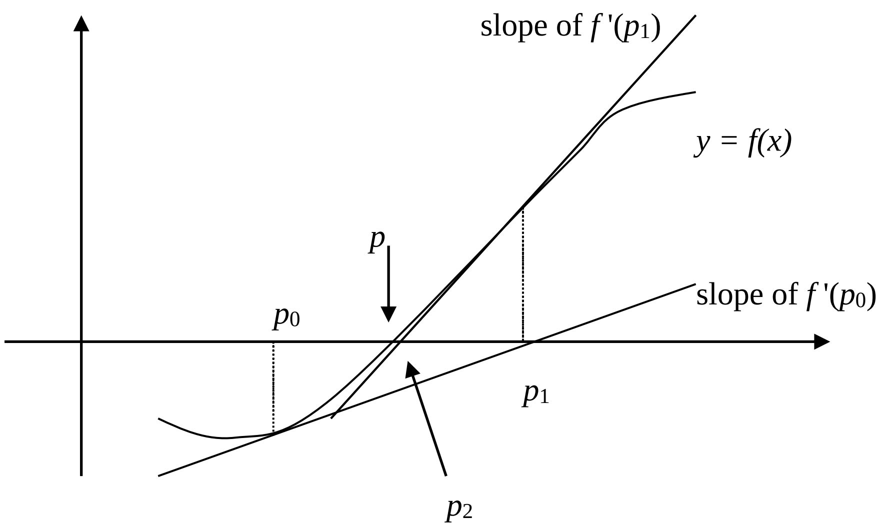

Figure 3: The graphical representation of the Newton-Raphson method

IN GENERAL

The approximation p_n to a root of $f(x)=0$ is computed from the approximation p_n using the equation

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Example 3: Find a root of $f(x)=x^3+4x^2-10=0$ starting with $p_0=1$ and tolerance 1×10^{-3} by *Newton's* method.

Iteration, n	p_n	$f(p_n)$
1	1.454545	1.540187
2	1.368900	0.060712
3		
4		
5		

4. CONVERGENCE ANALYSIS

Quadratic Convergence of Newton Method

In Example 3, the errors are computed and shown in the following table.[r is the actual root]

Iteration, n	p_n	$e_n = \lvert r-P_n\rvert$	e_{n+1}/e_n^2
1	1.454545454545	0.089315441131	0.46010699983
2	1.368900401070	0.003670387655	0.48893352379
3	1.365236600202	0.000006586788	0.49024483292
4	1.365230013435	0.000000000021	
5	1.365230013414	0	

We can see how fast the error becomes small. It is noted that once convergence starts to hold, the number of correct places in P_n approximately doubles on each iteration. This is characteristic of "quadratic convergent" method.

Theorem 3.1: Let *f* be twice continuously differentiable and $f(r)=0$. If $f'(r)\neq 0$, then Newton's Method is locally and quadratically convergent to *r*.

The error e_n at each step *n* satisfies $\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=M$ where $M=\left|\frac{f''(r)}{2f'(r)}\right|$.

The convergence rate is $e_{n+1}\approx Me_n^2$.

In Example 3, evaluating at *x*=1.36523 yields $M\approx 0.49$ which agrees with the error ratio in the right column of the table.

Linear Convergence of Newton Method

Theorem 3.1 does not say that Newton Method always converges quadratically. In the case where $f'(r)=0$, the error formula $M=\left|\frac{f''(r)}{2f'(r)}\right|$ does not work. This is a case for multiple roots.

Theorem 3.2: Assume that the $(m+1)$ times continuously differentiable function f on $[a,b]$ has a multiplicity m root at r. Then Newton method is linearly convergent to r, and error e_n at each step n satisfies $\lim_{n\to\infty}\frac{e_{n+1}}{e_n}=S$, where $S=\frac{m-1}{m}$. The convergence rate is $e_{n+1}\approx Se_n$

In this case, the slower rate puts Newton's Method in the same category as Bisection Method.

Example 4

$f(x)=\sin x+x^2\cos x-x^2-x$, $P_0=0.1$

$f'(x)=\cos x+2x\cos x-x^2\sin x-2x-1$

$f''(x)=-\sin x+2\cos x-4x\sin x-x^2\cos x-2$

$f'''(x)=-\cos x-6\sin x-6x\cos x+x^2\sin x$

$f(x),\ f'(x),$ and $f''(x)$ each evaluates to 0 at r=0.

$f'''(0)=-1$ so the root r=0 is a triple root, meaning the multiplicity m=3.

By theorem 3.2, Newton converge linearly with convergence proportionality $S=\frac{m-1}{m}=\frac{2}{3}$, and $e_{n+1}\approx\frac{2}{3}e_n$ which agrees with the error ratio in the right column of the table.

Iteration, n	p_n	$e_n=\lvert r-P_n\rvert$	e_{n+1}/e_n
1	0.1	0.1	0.69036
2	0.069036	0.069036	0.68464
3	0.047265	0.047265	0.67989
4	0.032135	0.032135	0.67615
5	0.021728	0.021728	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Convergence Analysis of Secant Method

Under the assumption that the Secant Method converges to r and $f'(r) \neq 0$, the following approximate error relationship holds.

$$e_{n+1} \approx \left|\frac{f''(r)}{2f'(r)}\right| e_n e_{n-1}$$

The convergence of the secant method to simple root is called superliner, meaning that it lies between linearly and quadratically convergent methods.

Iteration, n	p_n	$e_n = \lvert r - P_n \rvert$	$e_{n+1} / e_n e_{n-1}$
1	1.263158	0.102072	
2	1.338828	0.026402	0.514304
3	1.366616	0.001386	0.491895
4	1.365212	0.000018	0.490220
5	1.365230	$1.223*10^{-8}$	